Ergodic Theory

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1. Basic ideas

Ergodic theory is a mathematical endeavour which arose from the study of statistical mechanics by physicists in the latter half of the nineteenth century, as an ongoing attempt to derive the macroscopic, statistical laws of thermodynamics from deterministic microscopic behaviour. The basic concept which is studied in this mathematical discipline is that of the *measure preserving transformation.* Thus my first task is to create an image inside of your head, reader, of what we think of when we hear this collection of words. The most important word, *transformation*, indicates that we are dealing with a change, or movement, of a collection of basic (i.e. indistinguishable except for their names) objects, and the other two words, *measure preserving*, are intended to show that the sizes of subcollections of these objects do not change after the movement, or transformation, is applied.

1.1. A simple example

Here is a simple example. Consider three indistinguishable objects, placed in positions which we simply denote by a, b, and c respectively. Each of the three objects has the same *size*, where it is perhaps best to think of the size of an object as its weight; we generally call this non-negative number the *measure* of the object. The positions a, b, and c are usually called *points.* Now imagine the following movements taking place simultaneously:

the object at point a moves to point b, the object at point b moves to point c, and the object at point c moves to point a. Thus we have defined a measure preserving transformation, since after the movement each position is occupied with an object of the same size as before. The objects, of course, can now disappear from our discussion, since they are only distinguished by their positions and we can think of the measure of an object as a number attached to, or more generally a mass distribution over, the collection of positions; this is a typical mathematical ploy.

1.2. The measure preserving transformation
After this simple example, we jump to the attempt at creating a general picture in your head. A *measure preserving transformation* is defined as a collection of points (i.e. a *set* or a *space*) together with a mass distribution over this space of points, and a transformation assigning to each point of the space another (perhaps in some instances the same) point of the space, such that after simultaneous application of the transformation to each point of the space, the same mass distribution is observed.

Here we perhaps need to remark that our initial example was very simple, in that we were dealing with a finite set of points. The most interesting and natural situations deal with much larger sets of points, such as the interval of real numbers between 0 and 1, or more generally spaces whose 'points' are themselves collections of other objects, e.g. paths of particles or positions of sets of points. In these situations, it is more difficult to define the concept of mass distribution, and there is an entire branch of mathematics developed around the beginning of the twentieth century, called *measure theory*, which lays down the rules for mass distributions and their behaviour under transformations. A thorough knowledge of measure theory is indispensable for research in ergodic theory, although on an intuitive level the concept of mass distribution and mass transportation seems to be easily accessible to a general audience.

1.3. A more interesting example
Let me now try to fill out the abstract picture given above with a more interesting example, which was one of the motivations for the study of ergodic theory at its beginning in the nineteenth century. Imagine a box filled with a large number N of gas molecules (for example, air, or more simply, hydrogen). At any fixed time, we can visualize the situation in the box by writing down the exact position and velocity of each of the molecules in a (very long) vector $\vec{x}$ of $6N$ real numbers. (I have written 'visualize' because of the practical impossibility of carrying out such a description; the number N will be much too large in any reasonable situation.) Now imagine the space X of all possible vectors $\vec{x}$ such that their *energy*, which is simply a number we can calculate from the entries of the given vector $\vec{x}$ by a simple

formula which will not concern us further here, is a fixed number E. The mass distribution we want to consider over X is the natural uniform distribution with total mass one, and the movement is given by starting at time zero in the configuration given by $\vec{x}$ and then calculating the positions and velocities of each of the molecules at time one, putting them all together in another long vector which we denote by $T(\vec{x})$ and call the transformed point. The calculations can be done according to different rules, but let us suppose here that we are interested in the rules given by classical mechanics. The measure preserving transformation thus described is commonly known as 'gas in a box', and the preservation of mass was first proved by Liouville in the middle of the nineteenth century.

2. Questions of interest in ergodic theory

Our next task is to describe some of the basic questions of interest in the field of ergodic theory. From the second example it should be clear that one of the goals is to get away from very detailed, local investigations of the behaviour of individual molecules or points, as in this example it would be impossible to say very much. The simplest way to formulate this restriction is to realize that we wish to deal with successive movements, and in particular to try to describe the long-term behaviour after many many iterations of the measure preserving transformation. In the first example, things are quite clear; after two movements, a is at c, b at a, and c at b, and after three movements everyone is back to his starting spot and things repeat as before. In other words, this is a periodic transformation with period 3. The second example presents more difficulty, but just recently it has been shown by the Hungarian School (for identical molecules of a fixed size and so-called elastic collisions with each other and with the sides of the box) that except for a set of starting points having probability zero, the positions and velocities will come arbitrarily close to *any* given set of positions and velocities again and again as the movement is iterated, after a sufficient number of movements. Thus, with probability one, all of the molecules will eventually collect in the right half of the box (but not stay there), if we wait long enough! This 'inevitable suffocation', although mathematically sound, is also very interesting because it contradicts the second law of thermodynamics, although it has been deduced from the first principles of classical mechanics, the only acceptable physical principles on a microscopic level for a wide class of gas models and densities.

After the above detour into the world of physical interpretation, we now return to mathematics, with a discussion of some mathematical problems and a few results obtained in the past years in The Netherlands and elsewhere connected to ergodic theory. Below we treat three areas of interest: percolation, one-dependent processes, and interval dynamics. We shall try to exhibit the corresponding measure preserving transformation, but it will not be possible to arrive at a detailed understanding of the underlying connections and proofs. Most of the work discussed has been carried out at Delft University of Technology, and substantially supported by the NWO/SMC-grant 'Coding Problems in Ergodic Theory', as well as other local and national funding.

3. Percolation

3.1. Mathematical models of percolation

In mathematical models of percolation, the underlying measure preserving transformation is not temporal, but spatial. The simplest model runs as follows. Imagine a regular grid of interconnected pipes in three-dimensional space, and suppose that a certain percentage of the pipes are open, allowing liquid to flow through them, while the remaining pipes are blocked. We assume that this configuration has been obtained by some random mechanism, for instance an independent coin toss with the same coin for each pipe. The space X in this situation is the collection of all (infinite) prescriptions of which pipes are open and which are blocked, while the movement is a spatial movement of translation in a direction parallel to some of the pipes, by the length of one of the pipes. There are in this example actually six different directions, so we have six measure preserving transformations, coming in pairs which are inverses of each other. The mass distribution is described by the random pipe blocking mechanism. The basic idea of percolation theory, developed by physicists earlier but put on a sound mathematical basis in the 1950's, is that if only a small percentage of the pipes are blocked, then there will be paths stretching to infinity along which liquid can flow, but if the blocking percentage is large, then all of the liquid is localized and cannot escape from a finite region. Thus there should be a critical blocking percentage, below which these infinite open paths appear and above which there are no infinite paths. (See also figure 1.) The basic applications of these ideas are in the areas of oil exploration and spread of disease, and it is interesting to be able to prove that critical blocking percentages (generally called critical probabilities) exist, calculate them, and more generally describe the nature of the random picture of a realization of the process. Both geometric and measure theoretic issues are important here.

3.2. Recent results

In the short space available, it would be impossible to detail the connection between ergodic theory and percolation, and we must be content with the statement that this point of view has proven very fruitful in understanding and solving many of the open problems, and providing simple proofs of theorems previously believed to be very complicated in nature. I list shortly some of our results—the reader should be aware that the selection is made

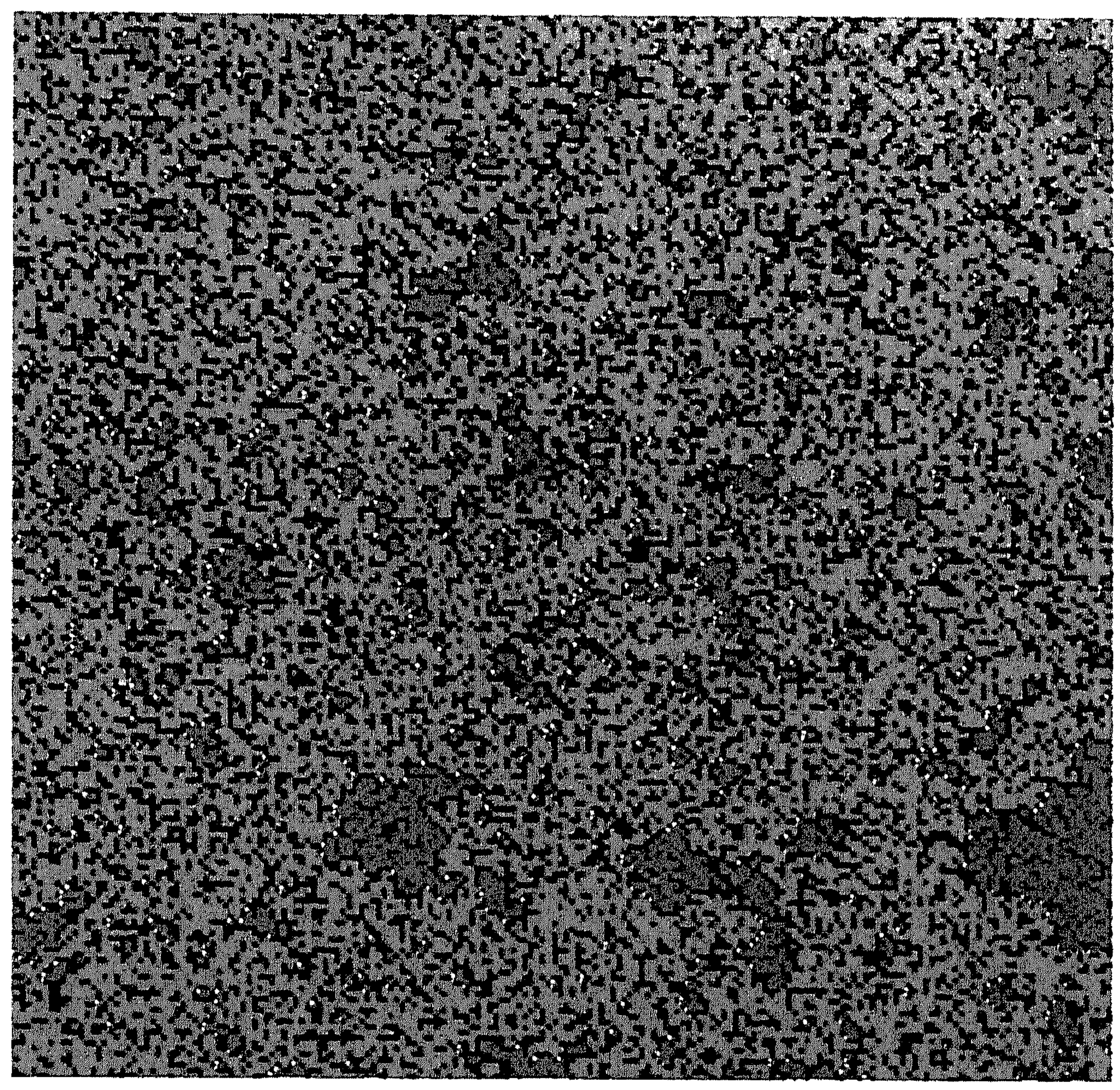

Figure 1. A computer realization of site percolation on the square lattice. Red is percolating.

to indicate the work carried out in the 1980's in The Netherlands, and is not a representative sample of the many interesting ideas in the field of percolation.

- The fundamental Van den Berg-Kesten-inequality, permitting calculation of several critical probabilities, and also of basic theoretical importance ([1]).
- The uniqueness of infinite clusters of open pipes and of densities of clusters in a large range of physically feasible percolation models ([2],[3]).
- Continuity results for percolation probability functions ([1]).
- Exact calculations for critical probabilities and percolation probability functions in circle percolation models ([5]).
- Connectivity and uniqueness in Mandelbrot percolation ([5]).

Most of these results do not directly use established methods of ergodic theory; instead, they raise substantial new questions concerning measure preserving transformations.

4. One-dependent processes

In ([6]), a long-standing conjecture concerning the existence of one-dependent processes which are not two-block factors of independent processes was settled, giving rise to a substantial collection of new one-dependent processes. Let me try to explain the idea behind one-dependent processes. First of all, it is easiest to think of a process as a doubly infinite sequence of two symbols, say 0 and 1, chosen in some random manner. For instance, suppose that for each element of the sequence we flip a coin which is labelled 0 on one side and 1 on the other (the coin need not be fair, but it should be the same coin for every element of the sequence). This gives rise to an independent (stationary) process. A generalization of independent processes, widely studied, is that of Markov processes, in which there are two coins, and the one flipped depends on the outcome of the flip in the preceding element of the sequence. Markov processes have the property that the future flips are independent of the past flips if one supposes the value of the present flip to be known. The *definition* of a one-dependent process is one in which the future is independent of the past, when no knowledge of the present is assumed. That is, the observer sees an event in the past, goes to sleep at the present and misses an observation, and then observes independence in the future with respect to what he saw in the past. It is easy to see that if we take any independent process (with perhaps more than two symbols, even more than a finite or countable number of states), and make another process by a function depending only on two successive observations with values 0 and 1, then this process is one-dependent. The conjecture was that every two-state one-dependent process arises in this manner, and ([6]) contains a large number of new one-dependent processes, which cannot arise in this manner. It is not easy to see how ergodic theory can be of help in this problem, and the underlying transformation is difficult to describe, and not even measure preserving. The methods indicate a connection to and a generalization of quantum probabilistic reasoning, which is not yet well understood and will certainly be the subject of further investigation.

5. Interval dynamics

One of the most interesting problems of ergodic theory is to establish existence of and to calculate the values of invariant measures for a given transformation of a space. In particular, one-dimensional transformations (maps of the unit interval to itself) received a large amount of attention in recent years. Thesis [4] treats existence of invariant measures for such transformations, and provided cornerstones for a number of subsequent results.

6. CLASSIFICATION

Ergodic theory provides an explanation for the apparent randomness observed in physically deterministic models. One can attempt to determine the different types of randomness by *classifying* measure preserving transformations. One of the most fruitful approaches to the classification problem is provided by the theory of finitary codes, developed by M. Smorodinsky and myself in the 1970's. At present, a thorough study of such classification of different types of pure randomness, both in classical and quantum descriptions, is being prepared.

7. INVARIANT MEASURES

As we have mentioned above, the problem of finding and making explicit invariant measures for a given transformation or transformations is central to ergodic theory. I cannot resist closing this essay with an interesting open problem, due to H. Furstenberg. Let S and T be the transformations of the unit interval defined by $Sx = 2x \bmod 1$ and $Tx = 3x \bmod 1$. Then the normalized Lebesgue measure (uniform distribution on the unit interval) is invariant under both S and T. Does there exist another continuous probability measure on the unit interval with this property?

8. CONCLUSION

In this short essay we have attempted to briefly describe the basic idea underlying the mathematical discipline of ergodic theory, to give a short description of its physical origins, and to describe summarily some aspects of work at Delft University of Technology using ergodic theory to answer fundamental questions inside the discipline and to contribute to problems in related fields. The essential reason for the wide range of application is the fundamental nature of the notion of a measure preserving transformation, together with its surprising complexity.

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